SOME CONSIDERATIONS ABOUT THE THERMAL ANALYSIS OF A THERMOELECTRIC LEG

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Abstract

This paper presents some considerations about the thermal analysis of a thermoelectric leg. The explicit expressions of the temperature and the heat flux along the leg are obtained for steady state and also transient state. The influence of the Thomson effect is investigated in the case of steady state heat transfer and the Laplace transform is used to solve the transient case.

Introduction

There is an increasing use of thermoelectric devices with success in many fields such as aerospace, automotive and building applications. On the other hand, the coupled effects involved in such system usually leads to complex modelling. In order to predict the performances of the device, few approaches can be driven to estimate the efficiency. Several methods could be used: experimental, numerical and semi-analytical. For the experimental ones, the device must already exist whereas numerical and semi-analytical methods could provide more or less realistic predictions.

In this paper, a semi-analytical method has been chosen in order to better understand the underlying physical phenomena and the contribution of the different effects. A thermal modeling of a thermoelectric leg is presented. The aim is to determine the expressions of the temperature within the thermoelement and also the heat fluxes. Indeed these two quantities are needed to determine the performance of the device by calculating the efficiency of the element or for instance by evaluating the COP. The steady-state and the transient cases are investigated. The Joule contribution is taken into account (introducing a source term in the heat transfer equation) but in a first approximation the Thomson coefficient is assumed to be equal to zero. Explicit expressions of the transient temperature and of the heat flux are then obtained.

Equations and semi analytical solutions

Consider a single thermoelectric leg of length l and cross-sectional area A. An electrical current $I=JA$ enters uniformly into the element. The one-dimensional energy balance [1,2] that describes the thermal behaviour of the leg is the following partial differential equation

$$\rho c_p \frac{\partial T(x,t)}{\partial t} = \lambda \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{J^2}{\sigma} \qquad (1)$$

The temperature is a function of the spatial variable x and the time t. The relevant material properties are the density ρ, the heat capacity c_p, the thermal conductivity λ, the electrical conductivity σ.

- **Steady-state case**

In the steady-state case, the classical boundary conditions are the following ones: the hot side of the leg is at absolute temperature T_H and the cold side at temperature T_C. The equation (1) becomes a classical ordinary differential equation. The temperature within the leg is then given by the analytical expression:

$$T(x) = T_C + \left(\frac{T_H - T_C}{L} + \frac{J^2 L}{2\sigma\lambda} \right) x - \frac{J^2}{2\sigma\lambda} x^2 \quad (2)$$

If the Thomson effect is taken into account, the expression of the temperature contains no quadratic terms but exponential terms:

$$T(x) = T_C + \frac{J}{\sigma\tau} x + \frac{T_H - T_C - \frac{J}{\sigma\tau} L}{\left(1 - \exp\left(\frac{\tau J}{\lambda} L\right)\right)} \left(1 - \exp\left(\frac{\tau J}{\lambda} x\right)\right) \quad (3)$$

The influence of the Thomson coefficient on the temperature profile into the leg is represented in the Figure 1 and the data used to perform the simulations are summarized in the Table1.

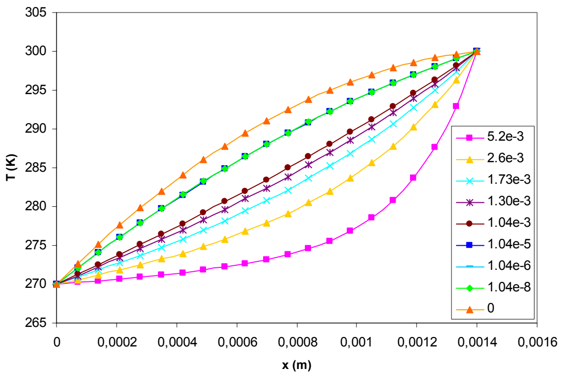

Figure 1: Influence of the Thomson coefficient on the temperature profile

T_C (K)	270	T_H (K)	300
L (m)	1.4e-3	A (m^2)	1.4e-6
I (A)	2	σ ($\Omega^{-1}m^{-1}$)	9.74e4
λ ($Wm^{-1}K^{-1}$)	1.7	α (VK^{-1})	2.07e-4

Table 1: Data used for simulations

It is interesting to have the analytical expression of the temperatures given by (2) and (3) because it makes it obvious which coefficients groups are important.

If there is no Thomson effect, an important parameter for the temperature distribution within the leg is $\xi = J^2/2\sigma\lambda$ (of course the difference between the hot and the cold temperatures plays also a significant role).

If the Thomson effect is taken into account, the important groups are: $\zeta_\tau = J/\sigma\tau$ and $\eta_\tau = \tau J/\lambda$. One notes that $\zeta_\tau \eta_\tau = 2\xi$.

It is although interesting to have the expression of the heat flux going through the thermoelement in order to have a complete modelling of the heat transfer within the leg.

Now, let express the heat flux which is a linear combination of the temperature and the derivative of the temperature:

$$\varphi = \alpha I T - \lambda A \frac{dT}{dx} \quad (4)$$

where α is the Seebeck coefficient.

The expression of the heat flux going through the leg is:

$$\varphi(x) = a_0 + a_1 x + a_2 x^2 \quad (5)$$

$$a_0 = \alpha I T_C - \frac{\lambda A}{L}(T_H - T_C) - \frac{I^2 L}{2\sigma A} \quad (5a)$$

$$a_1 = \alpha I \left(\frac{T_H - T_C}{L} + \frac{J^2 L}{2\sigma\lambda} \right) + \frac{AJ^2}{\sigma} \quad (5b)$$

$$a_2 = -\frac{\alpha I J^2}{2\sigma\lambda} \quad (5c)$$

If the Thomson effect is not neglected, the analytical expression of the heat flux is:

$$\varphi(x) = b_0 + b_1 x + b_2 \exp(\tau J x/\lambda) \quad (6)$$

$$b_0 = \alpha I (T_C + \Xi) - (\lambda A J/\sigma\tau) \quad (6a)$$

$$b_1 = \alpha I J/\sigma\tau \quad (6b)$$

$$b_2 = \alpha I \Xi + A \tau J \Xi \quad (6c)$$

$$\Xi = \left(T_H - T_C - \frac{J}{\sigma\tau}.L \right) \Big/ \left(1 - \exp\left(\frac{\tau J}{\lambda}.L \right) \right) \quad (6d)$$

The heat flux is plotted in the Figure 2, the entropy flux density in the Figure 3 without Thomson effect and also for a Thomson coefficient $\tau = 1.04e\text{-}4\ VK^{-1}$.

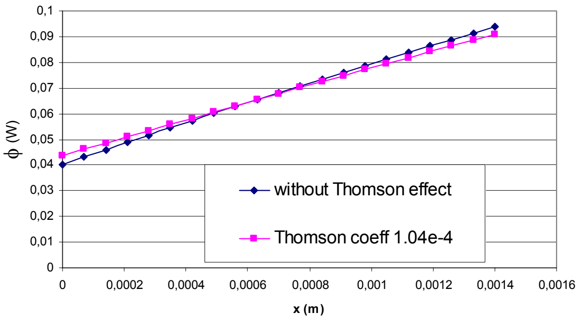

Figure 2: Heat flux along the TE leg

The entropy flux density Js =ϕ/AT is calculated via equations 3 and 5 (and respectively with equations 4 and 6 if the Thomson effect is taken into account).

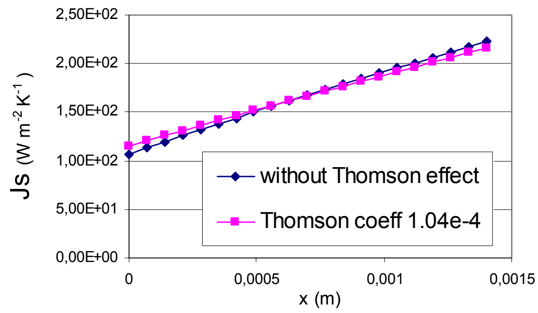

Figure 3: Entropy flux density along the leg

The entropy generated Sgen = d(Js)/dx is easily obtained from the entropy flux density (see Figure 4).

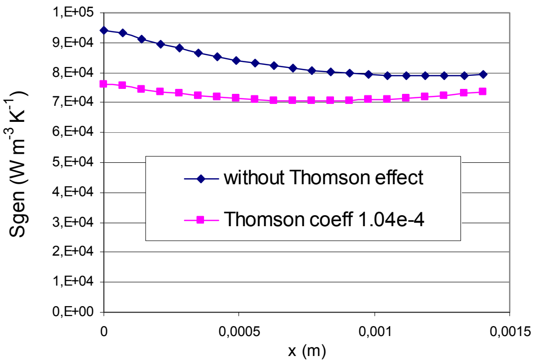

Figure 4: Entropy generated along the leg

- **Transient case**

For the transient case, let consider the equation (1) written in a more synthetic way:

$$\frac{\partial^2\theta(x,t)}{\partial x^2}-\frac{1}{a}\frac{\partial\theta(x,t)}{\partial t}=-\xi \qquad (7)$$

$$\theta = T - T_0\,, \qquad \xi=\frac{J^2}{\sigma\lambda} \qquad (8)$$

The typical property used in transient case is the thermal diffusivity *a*; the initial temperature of the leg is T_0.

To solve this partial differential equation, it is very convenient to apply the Laplace transform which transforms the partial differential equation into an ordinary differential equation in the Laplace domain. Let introduce *p* the Laplace variable and note:

$$L(f)=\bar{f}(p)=\int_0^{+\infty} f(x,t)\,\exp(-pt)\,dt \qquad (9)$$

The equation (7) becomes

$$\frac{d^2\bar{\theta}}{dx^2}-\frac{p}{a}\cdot\bar{\theta}=-\frac{\xi}{p} \qquad (10)$$

The solution of the equation (10) is the sum of the general solution of the homogeneous equation and a particular solution:

$$\bar{\theta}=C\exp\left(\sqrt{\frac{p}{a}}x\right)+D\exp\left(-\sqrt{\frac{p}{a}}x\right)+\frac{\xi\,a}{p^2} \qquad (11)$$

where C and D are to be determined by the boundary conditions. They depend on the Laplace variable *p* but not on the space variable *x*.
The boundary conditions are the following:

$$\bar{\theta}(x=0)=\bar{\theta}_0=0 \qquad (12a)$$

$$\bar{\theta}(x=L)=\bar{\theta}_L=\Delta/p \qquad (12b)$$

$$\Delta=\varepsilon(T_H-T_C) \qquad (12c)$$

with ε=+1 if heating and ε=–1 if cooling.

Considering equations (12) together with equation (11), it comes:

$$C=\frac{-\frac{\Delta}{p}+\frac{J^2a}{\sigma\lambda p^2}\left(1-\exp\left(-\sqrt{p/a}L\right)\right)}{\exp\left(-\sqrt{p/a}L\right)-\exp\left(\sqrt{p/a}L\right)} \qquad (13a)$$

$$D=\frac{-\frac{\Delta}{p}+\frac{J^2a}{\sigma\lambda p^2}\left(1-\exp\left(\sqrt{p/a}L\right)\right)}{\exp\left(\sqrt{p/a}L\right)-\exp\left(-\sqrt{p/a}L\right)} \qquad (13b)$$

Another explicit expression of the temperature within the leg is after rearranging the exponentials:

$$\bar{\theta} = \frac{1}{\sinh(s^* L)} \left\{ \frac{\Delta}{p} \sinh(s^* x) \right\} + \frac{J^2 a}{\sigma \lambda p^2} + \frac{\frac{J^2 a}{\sigma \lambda p^2} \{ \sinh(-s^* x) + \sinh(s^* (x - L)) \}}{\sinh(s^* L)} \quad (14)$$

with $s^* = \sqrt{p/a}$.

The expression of the temperature (14) is given in the Laplace space. The Stephest [3] or De Hoog [4] method is used to come back in the time domain. The transient temperature in the leg is computed with Matlab and plotted in the Figure 5. The value of the diffusivity chosen for the simulations is $a = 4.19501\text{e-}7\ \mathrm{m^2 s^{-1}}$.

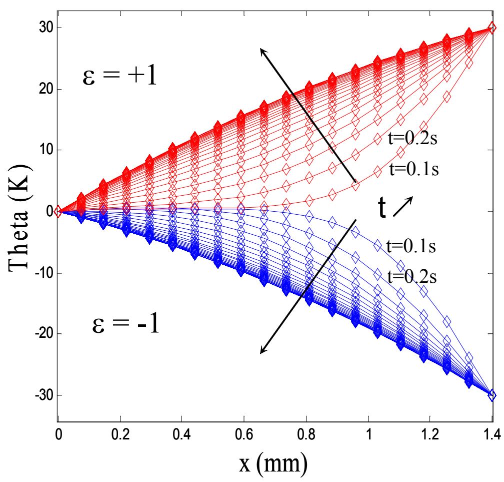

Figure 5: Temperature profiles within the leg at several times steps

The important parameters on the temperature are still Δ and ξ but there is another group appearing due to the transient state: it is the ratio of the Laplace variable over the thermal diffusivity i.e. p/a and also its square root and pL^2/a (one recognizes the obvious link with the characteristic time or Fourier time defined by $t^* = L^2/a$).

From these temperatures for transient state, it is possible to obtain the transient heat fluxes going through the leg and also the entropy heat fluxes or for instance a transient coefficient of performance.

The results proposed here have been obtained under specific boundary conditions, but if they changed, only the constants C and D have to be determined again.

Another practical way to solve the heat transfer equation given by equation (7) is to perform a quadrupole formulation of the problem. In that way, it provides a transfer matrix for the thermoelectric leg that linearly links the input temperature-heat flux column vector at the front side and the output vector at the back side. In that case, there is no need to determine explicitly the constants C and D.

Conclusion

A thermal analysis has been performed for a thermoelectric leg for steady and transient state. When the Thomson effect is neglected, the temperature repartition is a second order polynomial law whereas it is an exponential combined with a linear term if the Thomson coefficient is not equal to zero. Not taking the Thomson effect into account leads to an overestimation of the temperature. Nevertheless as soon as the Thomson coefficient is low enough, its influence on the heat flux and entropy heat flux is not so important.

To obtain analytical expressions of the temperature, the heat flux or entropy allows bringing to the fore the parameters (group of thermophysical and thermoelectrical properties) which play a significant role on the heat transfer.

The outlook is to perform the whole quadrupole formulation of the heat transfer in a thermoelectric leg and to investigate the influence of all the parameters on the temperature repartition and on the COP.

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